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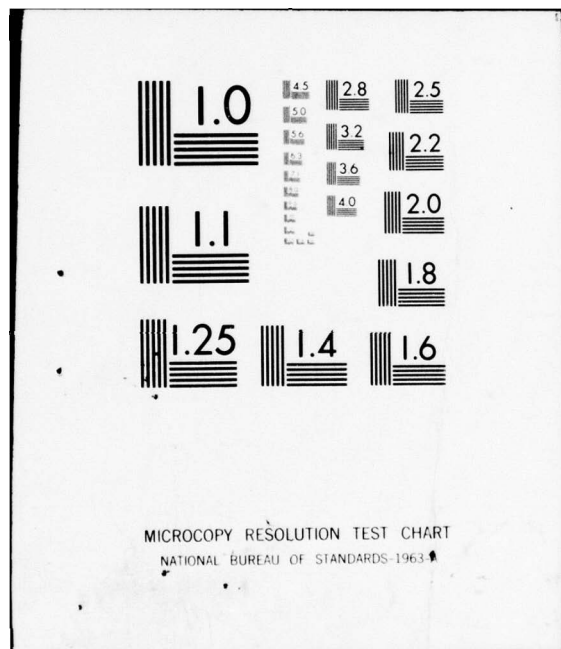
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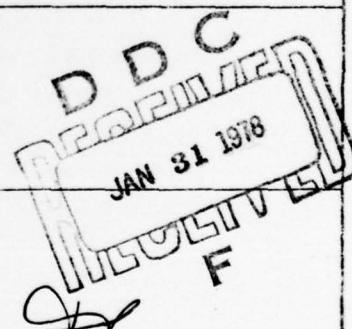
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Abstract

A minimum energy control problem is considered for commutative bilinear systems with and without terminal constraints. Optimal controls are shown to be constant vectors determined by the boundary conditions. Sufficient conditions are derived for uniqueness of the optimal control in the absence of a terminal constraint. A class of physical bilinear systems is discussed which possess the commutative property.

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I. Introduction

Consider the multi-input bilinear system

$$\dot{x} = \left[A + \sum_{i=1}^m B_i u_i \right] x \quad (1)$$

$$x(t_0) = x_0 \in R^n$$

where $(A, B_1 \dots B_m)$ are $n \times n$ constant matrices and $u = (u_1 \dots u_m)$ is a vector of control inputs with each component $u_i(t)$ assumed to be a square integrable function on every finite time interval, $t_0 \leq t \leq t_1$. A subclass of the system (1) is the "commutative bilinear system" in which each pair of matrices in the set $(A, B_1 \dots B_m)$ commute, i.e. $AB_i = B_i A$ and $B_i B_j = B_j B_i$ for all i, j . Sussmann [1] has shown that the attainable set for this subclass is closed relative to "bang-bang" control functions. Baras and Hampton [2] extended this result to a class of delayed commutative bilinear systems.

In this paper, a minimum energy control problem for commutative bilinear systems is studied both with and without endpoint constraints on the state. It is shown that a given terminal state $x(t_1) = x_1$ is reachable at some time $t_1 > t_0$ if and only if it is constant reachable, i.e. reachable using a constant input function. This suggests that the optimal control for a fixed terminal minimum energy regulation problem is simply a constant vector. In addition to verifying this supposition, the uniqueness of solutions is studied for the regulator problem without an end point constraint and a class of physical dynamic systems is delineated which possess the commutative property.

II. A Commutative Bilinear Physical System

The absolute velocity for the motion of a particle in a moving coordinate system is given by the vector equation (eg. see [3])

$$\underline{v} = \dot{\underline{R}} + (\dot{\underline{\rho}})_{\underline{r}} + \underline{\omega} \times \underline{\rho} \quad (2)$$

where $\dot{\underline{R}}$ is the absolute velocity of the origin of the moving coordinate system, $\underline{\rho}$ is the position vector of the particle relative to the moving origin, $\underline{\omega}$ is the angular rotation rate of the moving coordinate system and $(\dot{\underline{\rho}})_{\underline{r}}$ is the velocity of the particle relative to an observer fixed in the moving coordinate system. If the particle is associated with a target "T" and the moving origin with a pursuer "P", and if motion is confined to the (x,y) plane, then the kinematic equations for this motion can be derived as:

$$\begin{aligned} \dot{x} &= -v_T \sin \beta + u_P y & + \\ \dot{y} &= v_T \cos \beta - u_P x - v_P & (3) \\ \dot{\beta} &= u_T - u_P \end{aligned}$$

where (u_P, u_T) are the angular rates of the pursuer and target with respect to a nonrotating reference frame, (v_P, v_T) are the line speeds of the pursuer and target, (x, y) are the horizontal and vertical distances of the target relative to the pursuer, and β is the relative angle between the headings of the target and pursuer.

The kinematic equations (3) remain valid when all quantities (u_P, v_P, u_T, v_T) vary with time. These equations arise in a variety of pursuit-evasion type problems such as the "two car" and "homicidal chauffeur" games of Isaacs [4],

[†]In deriving (3) from (2) a unit triad $(\underline{i}, \underline{j}, \underline{k})$ is defined at P so that $\underline{\rho} = x\underline{i} + y\underline{j}$, $\dot{\underline{R}} = v_P \underline{j}$ and $\underline{\omega} = u_P \underline{k}$ at all times.

aerial combat between two aircraft [5] and a missile intercept problem [6]. In order to associate a commutative bilinear control system with (3) we assume that the pursuer can manipulate both its speed v_p and its angular acceleration u_p , and that $u_p(t)$ vanishes only at points of measure zero over the time interval of interest. Then there is no loss of generality in postulating the existence of a scalar valued function $\gamma(t)$ such that

$$v_p(t) = \gamma(t)u_p(t). \quad (4)$$

Defining state variables $x_1 = x$, $x_2 = y$ and $x_3 = \beta$ for (3), and introducing three auxiliary states according to $x_4 = \sin x_3$, $x_5 = \cos x_3$ and $x_6 = 1$, (3) in combination with (4) is seen to be equivalent to the bilinear system

$$\dot{x} = Ax + Bxu \quad (5-a)$$

$$x(t_0) = \text{Col}[x_1(t_0), x_2(t_0), x_3(t_0), \sin x_3(t_0), \cos x_3(t_0), 1]$$

$$A = \begin{pmatrix} 0 & 0 & 0 & -v_T & 0 & 0 \\ 0 & 0 & 0 & 0 & v_T & 0 \\ 0 & 0 & 0 & 0 & 0 & u_T \\ 0 & 0 & 0 & 0 & u_T & 0 \\ 0 & 0 & 0 & -u_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (5-b)$$

in which $u = u_p$ is regarded as the primary control variable. It can be readily verified that A and B commute for all (u_T, v_T, γ) , i.e.

$$AB = BA = \begin{pmatrix} 0 & 0 & 0 & 0 & v_T & 0 \\ 0 & 0 & 0 & v_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_T & 0 & 0 \\ 0 & 0 & 0 & 0 & u_T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and, moreover, this commutative property is independent of $\gamma(t) = v_p(t)/u_p(t)$.

An important problem for (5) is the determination of feasible controls for the intercept condition:

$$x_1(t_1) = x_2(t_1) = 0 \quad \text{for some } t_1 > t_0, \quad (6)$$

given the initial condition $x(t_0) = x_0$ and the target parameters (u_T, v_T) . If some leeway is allowed for the intercept angle $x_3(t_1) = \beta_1$ and if γ is regarded as a parameter subservient to the terminal constraint (6), then a solution exists for this problem in which the control is given by the constant function

$$u^* = u_T + \frac{x_3(t_0) - \beta_1}{t_1 - t_0}, \quad t_0 \leq t \leq t_1. \quad (7)$$

Integral to this solution is the fact that the parameters (β_1, γ) are easily obtained as solutions to a certain transcendental equation after which t_1 is given explicitly as a function of (β_1, γ) and the initial conditions. The details of this solution are given in Section IV of [7], along with a closed form least squares estimate of the parameters $(u_T, v_T, x_3(t_0))$, as part of a proposed feedback control law for a missile intercept system.

The question arises as to the interpretation of the control (7) which drives the commutative bilinear system (5) from an arbitrary initial state to the terminal state (6) in some time interval $t_0 \leq t \leq t_1$. The results of the next section will demonstrate that this simple control, which is relatively easy to compute and implement, is optimal in the minimum control energy sense.

III. A Minimum Energy Control Problem

The following definition of reachability is pertinent since we are dealing with a nonlinear control system.

Definition: The state vectors (x_0, x_1) constitute a reachable pair for the bilinear system (1) if there exists a square integrable control signal $u(t)$ which accomplishes the transition from $x_0 = x(t_0)$ to $x_1 = x(t_1)$ over some finite time interval $t_0 \leq t \leq t_1$. Likewise, the pair (x_0, x_1) is said to be constant reachable if it is reachable for a constant control signal.

In the case of a time invariant commutative bilinear system in which the matrices $(A, B_1 \dots B_m)$ commute pair-wise, the solution to (1) corresponding to any square integrable control signal $u(t)$ on $t_0 \leq t \leq t_1$ can be written as (see Sussmann [1])

$$x(t_1) = e^{A(t_1-t_0)} \prod_{i=1}^m \phi_i(t_1, t_0) x_0 \quad (8)$$

where e^{At} and $\phi_i(t, t_0)$ are the state transition matrices for the nxn matrices A and $B_i u_i(t)$, respectively.[†] The following result is then easily obtained.

Assertion A pair (x_0, x_1) for a time invariant commutative bilinear system is reachable if and only if it is constant reachable.

Proof: The sufficiency part of the proof is true by definition. The necessity part follows upon defining a constant control signal according to the average value

$$u_c = \frac{1}{t_1-t_0} \int_{t_0}^{t_1} u(t) dt . \quad (9)$$

If the pair (x_0, x_1) is reachable using a variable control $u(t)$ on $t_0 \leq t \leq t_1$, then (8) and (9) imply

[†]Note that $\phi_i(t, t_0) = \exp[B_i \int_{t_0}^t u_i(\tau) d\tau]$ owing to the special form of the time varying matrix $B_i u_i(t)$.

$$x_1 = e^{A(t_1-t_0)} \prod_{i=1}^m e^{B_i u_{c_i}(t_1-t_0)} x_0 \quad (10)$$

which is just the state response at time t_1 using the constant control (9). This verifies the Assertion.

The above result can be slightly generalized to a class of time varying commutative bilinear systems of the form

$$\dot{x}(t) = [A(t) + \sum_{i=1}^m B_i \alpha_i(t) u_i(t)] x(t)$$

where the matrices $(A(t), B_1, \dots, B_m)$ commute pair-wise for all t and (x_0, x_1) is a reachable pair corresponding to a certain control $u(t)$ on (t_0, t_1) . In this case the pair (x_0, x_1) is reachable using the constant control

$$u_{c_i} = \left[\int_{t_0}^{t_1} \alpha_i(t) dt \right]^{-1} \int_{t_0}^{t_1} \alpha_i(t) u_i(t) dt,$$

assuming that the given scalar functions $\alpha_i(t)$ are such that their integrals do not vanish over the time interval (t_0, t_1) .

A. Control Without Terminal Constraints

Consider the quadratic cost over a fixed time interval

$$J_1(u) = x'(t_1) F x(t_1) + \int_{t_0}^{t_1} u'(t) R u(t) dt \quad (11)$$

where F and R are symmetric nonnegative definite constant weighting matrices with $R > 0$. The optimal controls for (1) which minimize (11) must satisfy the following canonical equations from the maximum principle:

$$\dot{x}^* = [A + \sum_{i=1}^m B_i u_i^*] x^*, \quad x^*(t_0) = x_0$$

$$\dot{p}^* = -[A' + \sum_{i=1}^m B_i' u_i^*] p^*$$

$$p^*(t_1) = -F x^*(t_1)$$

$$u^* = \frac{1}{2} R^{-1} \begin{bmatrix} x^{*'} B_1' p^* \\ \vdots \\ x^{*'} B_m' p^* \end{bmatrix}.$$

Assuming commutativity between pairs of matrices in the set $(A, B_1 \dots B_m)$ it can be verified directly that $\frac{du^*}{dt}(t) = 0$ for all $t \in (t_0, t_1)$. Putting $t = t_1$ in $u^*(t)$, utilizing the boundary condition for $p^*(t_1)$ and the relation (10), it follows that the constant vector u^* , if it exists, must satisfy the nonlinear transcendental equation

$$u^* = -\frac{1}{2} R^{-1} \begin{bmatrix} x_0' e^{A'(t_1-t_0)} \prod_{i=1}^m e^{B_i' u_i^*(t_1-t_0)} B_{1F}' \prod_{i=1}^m e^{B_i u_i^*(t_1-t_0)} e^{A(t_1-t_0)} x_0 \\ \vdots \\ x_0' e^{A'(t_1-t_0)} \prod_{i=1}^m e^{B_i' u_i^*(t_1-t_0)} B_{mF}' \prod_{i=1}^m e^{B_i u_i^*(t_1-t_0)} e^{A(t_1-t_0)} x_0 \end{bmatrix} \quad (12)$$

$$\underline{A} - G(u^*).$$

The existence of optimal controls can be established independently of the commutativity condition through the study of the properties of the attainable set for (1) and (11) corresponding to all square integrable input functions. The details of this study can be found in Chapter 3 of [8]. Therefore, any conclusions based on the necessary conditions of the maximum principle are valid. These results are summarized in the following theorem together with a sufficient condition for uniqueness.

Theorem 1: The optimal controls for a time invariant commutative bilinear system with the quadratic cost (11) are constant vectors u^* satisfying the transcendental equation (12). The vector u^* is unique if the following matrix valued function $Z(v_0) = (Z_{ij})$ is nonnegative definite for all $v_0 \in R^n$:

$$Z_{ij} = v_0' [B_j' B_i' F + B_i' F B_j] v_0 \quad (13)$$

$$i, j = 1, 2, \dots, m.$$

For a single input commutative bilinear system ($m = 1$), uniqueness is guaranteed for the simpler nonnegativity condition:

$$B_1'^2 F + B_1' F B_1 \geq 0.$$

Proof Since the characterization of u^* by (12) has already been established, we consider the uniqueness of solutions to $u^* + G(u^*) = 0$, where the operator G is defined by the right side of (12). From the theory of monotone operators (see Minty [9]) it is known that the equation $u + G(u) = 0$ has a unique solution if G is a monotonically increasing map. If G is Fréchet differentiable, a sufficient condition for this is that the Fréchet derivative operator $G'(u)$ be nonnegative definite. By direct computation the Fréchet differential of G at u^* with increment h is:

$$dG(u^*; h) = \frac{1}{2} R^{-1} \tilde{Z}(x_0) \cdot h, \quad h \in R^m$$

where

$$\tilde{Z}_{ij} = x_0' e^{A'(t_1 - t_0)} \prod_{k=1}^m e^{B_k' u_k^*(t_1 - t_0)} [B_j' B_i' F + B_i' F B_j] \prod_{k=1}^m e^{B_k u_k^*(t_1 - t_0)} e^{A(t_1 - t_0)} x_0.$$

By hypothesis, the matrix $Z(v_0)$ defined by (13) is nonnegative definite, which implies that the matrix $\tilde{Z}(x_0)$ above is also nonnegative definite. In turn, this assures the monotonicity of G (see Vainberg [10]).

If the more general quadratic cost

$$J(u) = x'(t_1)Fx(t_1) + \int_{t_0}^{t_1} [x'(t)Qx(t) + u'(t)Ru(t)]dt$$

is prescribed in order to weight the state trajectory $x(t)$ through the symmetric nonnegative definite matrix Q , then the optimal controls satisfy

$$\dot{x}^* = [A + \sum_{i=1}^m B_i u_i^*]x^*$$

$$x^*(t_0) = x_0$$

$$\dot{p}^* = -[A' + \sum_{i=1}^m B_i' u_i^*]p^* + 2Qx^*$$

$$p^*(t_1) = -Fx^*(t_1)$$

$$u^* = \frac{1}{2} R^{-1} \begin{pmatrix} x^{*'} B_1 p^* \\ \vdots \\ x^{*'} B_m p^* \end{pmatrix}.$$

In this case, the control derivative, $\frac{du^*}{dt}$, becomes (again, using commutativity)

$$\frac{du^*}{dt}(t) = R^{-1} \begin{pmatrix} x^{*'}(t)QB_1x^*(t) \\ \vdots \\ x^{*'}(t)QB_mx^*(t) \end{pmatrix} \neq 0.$$

Although this control represents a simple pure integration on quadratic operations involving the state, it is considerably more difficult to compute since the boundary condition is not characterized by an explicitly defined algebraic equation as in (12).

B. Control With Terminal Constraints

We now consider the minimum control energy problem for a commutative bilinear system subject to a fixed terminal state $x(t_1) = x_1$.

Theorem 2: Given a time invariant commutative bilinear system and a reachable pair (x_0, x_1) , the optimal controls for the cost function

$$J_2(u) = \int_{t_0}^{t_1} u'(t)Ru(t)dt, \quad R > 0 \quad (14)$$

are constant vectors u^* satisfying the transcendental equation

$$x_1 = e^{A(t_1-t_0)} \prod_{i=1}^m e^{B_i u_i^*(t_1-t_0)} x_0. \quad (15)$$

Proof: From the Assertion it is known that if the pair (x_0, x_1) is reachable using some control $u(t)$, $t_0 \leq t \leq t_1$, then it is reachable using the constant control given by (9). Therefore, comparing the costs associated with these two control functions

$$U_1 = \int_{t_0}^{t_1} u_c' R u_c dt = u_c' R u_c (t_1 - t_0)$$

$$U_2 = \int_{t_0}^{t_1} u'(t)Ru(t)dt,$$

it follows from Hölder's inequality applied to relation (9) that $U_1 \leq U_2$ with equality if and only if $u(t) =$ a constant vector on (t_0, t_1) . Equation (15) is simply (10) rewritten. This proves the theorem.

Brockett [11] obtained a (non-constant) solution to the minimum control energy problem with a fixed terminal state in the case of $n \times n$ matrix state commutative bilinear systems with $\det X \neq 0$. By contrast, the solution here to the vector state case has been shown to be simply a constant control vector.

The following simple example illustrates the nonuniqueness of optimal controls for a minimum energy problem with a fixed terminal constraint.

Example: Given the commutative bilinear second order system

$$\begin{aligned}\dot{x}_1 &= x_1 + ux_2, & x_1(0) &= 0 \\ \dot{x}_2 &= x_2 - ux_1, & x_2(0) &= 1\end{aligned}$$

together with the cost to be minimized and the terminal constraint:

$$J(u) = \int_0^1 u^2(t) dt, \quad \begin{aligned}x_1(1) &= 0 \\ x_2(1) &= -e.\end{aligned}$$

It can be seen that (A, B_1) commute and that $(x(0), x(1))$ is a reachable pair. Hence, by Theorem 2 the optimal controls satisfy (15) applied to this example:

$$\begin{aligned}0 &= e \sin u^* \\ -e &= e \cos u^*.\end{aligned}$$

Solving for u^* we obtain $u^* = k\pi$, $k = \pm 1, \pm 3, \dots$, and in order that J be minimized it follows that $u_1^* = \pi$ and $u_2^* = -\pi$ are both optimal.

IV. Concluding Remarks

Although commutative bilinear systems form a very small subset of bilinear systems, there nevertheless exists an important class of physical systems for which the condition of commutativity is upheld. The minimum energy controls for this class relative to the costs (11) and (14) have particularly simple

forms as constant functions determined by the boundary conditions. Although these controls are often nonunique, any lack of uniqueness can possibly be exploited in other ways. For example, the intercept angle $\beta_1 = x_3(t_1)$ in the missile intercept problem discussed briefly in Section II can be traded off against the scaling parameter γ in obtaining physically meaningful solutions for the pursuer speed and acceleration (v_p, u_p) , as shown by the simulations carried out in [7].

In terms of general bilinear systems, where $\Delta = AB - BA \neq 0$, the relatively simple solution for the commutative case might be used as a nominal or generating solution for obtaining minimum energy controls in a power series expansion of the commutator Δ . These considerations remain to be explored.

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